

PARABOLIC BMO ESTIMATES FOR PSEUDO-DIFFERENTIAL OPERATORS OF ARBITRARY ORDER

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ABSTRACT. In this article we prove the BMO- L_∞ estimate

$$\|(-\Delta)^{\gamma/2}u\|_{BMO(\mathbf{R}^{d+1})} \leq N \left\| \frac{\partial}{\partial t} u - A(t)u \right\|_{L_\infty(\mathbf{R}^{d+1})}, \quad \forall u \in C_c^\infty(\mathbf{R}^{d+1})$$

for a wide class of pseudo-differential operators $A(t)$ of order $\gamma \in (0, \infty)$. The coefficients of $A(t)$ are assumed to be merely measurable in time variable. As an application to the equation

$$\frac{\partial}{\partial t} u = A(t)u + f, \quad t \in \mathbf{R}$$

we prove that for any $u \in C_c^\infty(\mathbf{R}^{d+1})$

$$\|u_t\|_{L_p(\mathbf{R}^{d+1})} + \|(-\Delta)^{\gamma/2}u\|_{L_p(\mathbf{R}^{d+1})} \leq N \|u_t - A(t)u\|_{L_p(\mathbf{R}^{d+1})},$$

where $p \in (1, \infty)$ and the constant N is independent of u .

1. INTRODUCTION

It is a classical result that if a second-order operator $A(t)u = a^{ij}(t)u_{x^i x^j}$ fulfills the uniform ellipticity

$$\delta|\xi|^2 \leq a^{ij}(t)\xi^i \xi^j \leq \delta^{-1}|\xi|^2, \quad \delta > 0$$

then it holds that for any $p > 1$ and $u \in C_c^\infty(\mathbf{R}^{d+1})$

$$\|\Delta u\|_{L_p(\mathbf{R}^{d+1})} \leq c(\delta, p) \|u_t - A(t)u\|_{L_p(\mathbf{R}^{d+1})}. \quad (1.1)$$

If $a^{ij}(t)$ are smooth enough, then (1.1) can be obtained by using the multiplier theory. The classical multiplier theory is not applicable if $a^{ij}(t)$ are merely measurable in t . In this case one can rely on either Carlderón-Zygmund theory (see [7]) or the approach based on the sharp function estimate of Δu (see [8]).

In this article we extend (1.1) to a wide class of arbitrary order pseudo-differential operators $A(t)$ with measurable coefficients based on a BMO- L_∞ estimate. More precisely we prove

$$\|(-\Delta)^{\gamma/2}u\|_{BMO(\mathbf{R}^{d+1})} \leq N \|u_t - A(t)u\|_{L_\infty(\mathbf{R}^{d+1})}, \quad \forall u \in C_c^\infty(\mathbf{R}^{d+1}) \quad (1.2)$$

under the condition that there exist constants $\nu, \gamma > 0$ so that for the symbol $\psi(t, \xi)$ of $A(t)$ (i.e. $\mathcal{F}(A(t)u)(\xi) = \psi(t, \xi)\mathcal{F}(u)(\xi)$) it holds that

$$\Re[\psi(t, \xi)] \leq -\nu|\xi|^\gamma, \quad \forall \xi \in \mathbf{R}^d \setminus \{0\} \quad (1.3)$$

and for any multi-index $|\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1$

$$|D^\alpha \psi(t, \xi)| \leq \nu^{-1}|\xi|^{\gamma-|\alpha|}, \quad \forall \xi \in \mathbf{R}^d \setminus \{0\}. \quad (1.4)$$

2010 *Mathematics Subject Classification.* 35S10, 35K30, 35B45, 35B05.

This work was supported by Samsung Science and Technology Foundation under Project Number SSTF-BA1401-02.

Based on the Marcinkiewicz's interpolation theorem and (1.2) we prove a generalization of (1.1), that is

$$\|u_t\|_{L_p(\mathbb{R}^{d+1})} + \|(-\Delta)^{\gamma/2}u\|_{L_p(\mathbb{R}^{d+1})} \leq N\|u_t - A(t)u\|_{L_p(\mathbb{R}^{d+1})}, \quad p > 1. \quad (1.5)$$

Using (1.5) one can obtain the unique solvability of the Cauchy problem

$$u_t = A(t)u + f, \quad t > 0; \quad u(0, \cdot) = 0$$

in an appropriate L_p -space.

Here are some examples of operators $A(t)$ satisfying conditions (1.3) and (1.4). If $A(t) = (-1)^{m-1} \sum_{|\alpha|=|\beta|=m} a^{\alpha\beta}(t) D^{\alpha+\beta}$ is a $2m$ -order differential operator then the symbol $\psi(t, \xi) = (-1)^m \sum_{|\alpha|=|\beta|=m} a^{\alpha\beta}(t) \xi^\alpha \xi^\beta$ satisfies (1.3) and (1.4) if $a^{\alpha\beta}(t)$ are bounded complex-valued measurable functions satisfying

$$\nu|\xi|^{2m} \leq \sum_{|\alpha|=|\beta|=m} \xi^\alpha \xi^\beta \Re [a^{\alpha\beta}(t)].$$

Our results cover the operators of the type

$$A(t)u = \int_{\mathbb{R}^d} \left(u(t, x+y) - u(t, x) - \chi(y)(u(t, x), y) \right) m(t, y) \frac{dy}{|y|^{d+\gamma}}$$

where $\chi(y) = I_{\gamma>1} + I_{\gamma=1}I_{|y|\leq 1}$ and $m(t, y)$ is a nonnegative measurable function satisfying appropriate conditions. See Section 6 for details and further examples. The issue regarding the compositions and powers of operators is also discussed in Section 6. In particular, for any operators $A_1(t)$ and $A_2(t)$ satisfying the prescribed conditions and constants $a, b > 0$, the operator $C(t) = -(-A_1)^a(-A_2)^b$ satisfies the conditions if for instance the symbols of $A_i(t)$ are real-valued.

Actually in this article we prove a generalized version of (1.2). We introduce an optimal condition on the kernel $K(t, s, x)$ (see Assumptions 2.1 and 2.2) so that the inequality

$$\left\| \int_{-\infty}^t \int_{\mathbb{R}^d} K(t, s, x-y) f(s, y) dy ds \right\|_{BMO(\mathbb{R}^{d+1})} \leq N \|f\|_{L_\infty(\mathbb{R}^{d+1})} \quad (1.6)$$

holds for any $f \in C_c^\infty(\mathbb{R}^{d+1})$ with constant N independent of f . It turns out that if $A(t)$ is an operator with the symbol $\psi(t, \xi)$ satisfying (1.3) and (1.4) then the kernel $K(t, s, x)$ related to the formula

$$(-\Delta)^{\gamma/2}u = \int_{-\infty}^t \int_{\mathbb{R}^d} K(t, s, x-y) f(s, y) dy ds, \quad f := u_t - A(t)u$$

satisfies our restrictions on the kernel, that is Assumptions 2.1 and 2.2.

Below is a short description on related works. In the setting of elliptic equations, the BMO- L^∞ estimate

$$\|K * f\|_{BMO(\mathbb{R}^d)} \leq N \|f\|_{L_\infty(\mathbb{R}^d)} \quad (1.7)$$

has been well studied with Calderón-Zygmund kernel K . See, for instance, [6]. It seems that the tools used in the literature to prove (1.7) are not efficient for parabolic equations. Beyond BMO- L^∞ estimate, when it comes to elliptic equations, BMO-BMO type estimates have been obtained in quite general setting (see, for instance, [1], [2], and [3]). However, to the best of our knowledge, there is no

BMO- L^∞ or BMO-BMO type estimate for parabolic equations. We only mention that the sharp function estimate of the type

$$(A(t)u)^\sharp(t, x) \leq \varepsilon [\mathbb{M}(A(t)u)^2]^{1/2}(t, x) + N(\varepsilon) [\mathbb{M}(u_t - A(t)u)^2]^{1/2}(t, x), \quad \varepsilon > 0$$

for parabolic equations is introduced e.g. in [8] (second order) and [4] ($2m$ -order, $m \in \mathbb{N}$). Here h^\sharp and $\mathbb{M}h$ represent the sharp function and maximal function of h respectively.

To prove (1.6), in place of the duality property of Hardy space H^1 typically used in the literature to prove (1.7), we employ only direct computations on the basis of properties of kernels.

Finally we introduce some notation used in the article. As usual \mathbf{R}^d stands for the Euclidean space of points $x = (x^1, \dots, x^d)$, $B_r(x) := \{y \in \mathbf{R}^d : |x - y| < r\}$ and $B_r := B_r(0)$. For multi-indices $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \{0, 1, 2, \dots\}$, $x \in \mathbf{R}^d$, and functions $u(x)$ we set

$$u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \quad D^\alpha u = D_1^{\alpha_1} \cdot \dots \cdot D_d^{\alpha_d} u,$$

$$x^\alpha = (x^1)^{\alpha_1} (x^2)^{\alpha_2} \dots (x^d)^{\alpha_d}, \quad |\alpha| = \alpha_1 + \dots + \alpha_d.$$

We also use D_x^m to denote a partial derivative of order m with respect to x . For an open set $U \subset \mathbf{R}^d$ and a nonnegative integer n , we write $u \in C^n(U)$ if u is n times continuously differentiable in U . By $C_c^\infty(U)$ we denote the set of infinitely differentiable functions with compact support in U . The standard L_p -space on U with Lebesgue measure is denoted by $L_p(U)$. We use “:=” to denote a definition. $[a]$ is the biggest integer which is less than or equal to a . By \mathcal{F} and \mathcal{F}^{-1} we denote the d -dimensional Fourier transform and the inverse Fourier transform, respectively. That is, $\mathcal{F}(f)(\xi) := \int_{\mathbf{R}^d} e^{-ix \cdot \xi} f(x) dx$ and $\mathcal{F}^{-1}(f)(x) := \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{i\xi \cdot x} f(\xi) d\xi$. For a Borel set $X \subset \mathbf{R}^d$, we use $|X|$ to denote its Lebesgue measure and by $I_X(x)$ we denote the indicator of X .

2. MAIN RESULTS

Fix $\gamma > 0$ throughout this article. For a locally integrable function h on \mathbf{R}^{d+1} , we define the BMO semi-norm of h on \mathbf{R}^{d+1} as follows :

$$\|h\|_{BMO(\mathbf{R}^{d+1})} = \sup_Q \frac{1}{|Q|} \int_Q |h(r, z) - h_Q| \, dr dz,$$

where $f_Q := \frac{1}{|Q|} \int_Q f(r, z) \, dr dz$ and the sup is taken all Q of the type

$$Q = Q_c(t_0, x_0) := (t_0 - c^\gamma, t_0 + c^\gamma) \times B_c(x_0), \quad c > 0, (t_0, x_0) \in \mathbf{R}^{d+1}.$$

Let K be a measurable function defined on \mathbf{R}^{d+2} so that $K(t, s, \cdot)$ is integrable for each $s < t$. Denote

$$\hat{K}(t, s, \xi) = \mathcal{F}(K(t, s, \cdot))(\xi),$$

where \mathcal{F} denotes the Fourier transform on \mathbf{R}^d .

Assumption 2.1. *There exists a measurable function H on \mathbf{R}^{d+1} such that for all $t > s$ and $\xi \in \mathbf{R}^d$,*

$$|\hat{K}(t, s, \xi)| \leq H(t - s, \xi) \tag{2.1}$$

and

$$\sup_{\xi} \int_0^{\infty} H(t, \xi) dt < \infty. \quad (2.2)$$

Assumption 2.2. *There exists a nondecreasing function $\varphi(t) : (0, \infty) \rightarrow [0, \infty)$ such that*

(i) *for any $s > r$ and $c > 0$,*

$$\int_r^s \int_{|z| \geq c} |K(s, \tau, z)| \, dz d\tau \leq \varphi((s-r)c^{-\gamma}); \quad (2.3)$$

(ii) *for any $s > r > a$,*

$$\int_{-\infty}^a \int_{\mathbf{R}^d} |K(s, \tau, z) - K(r, \tau, z)| \, dz d\tau \leq \varphi((s-r)(r-a)^{-1}); \quad (2.4)$$

(iii) *for any $s > a$ and $h \in \mathbf{R}^d$,*

$$\int_{-\infty}^a \int_{\mathbf{R}^d} |K(s, \tau, z+h) - K(s, \tau, z)| \, dz d\tau \leq \varphi(|h|(s-a)^{-1/\gamma}). \quad (2.5)$$

Note that

$$\begin{aligned} & \int_{-\infty}^a \int_{\mathbf{R}^d} |K(s, \tau, z+h) - K(s, \tau, z)| \, dz d\tau \\ &= \int_{s-a}^{\infty} \int_{\mathbf{R}^d} |K(s, s-\tau, z+h) - K(s, \tau, z)| \, dz d\tau. \end{aligned}$$

Thus, if $K(s, \tau, z) = K(s-\tau, z)$ then (2.5) is equivalent to

$$\int_b^{\infty} \int_{\mathbf{R}^d} |K(\tau, z+h) - K(\tau, z)| \, dz d\tau \leq \varphi(|h|b^{-1/\gamma}).$$

For a function f on \mathbf{R}^{d+1} , denote

$$\mathcal{G}f(t, x) := \int_{-\infty}^t K(t, s, \cdot) * f(s, \cdot)(x) \, ds. \quad (2.6)$$

Remark 2.3. If f has compact support and is regular enough with respect x , then $\mathcal{G}f$ is well defined. For instance, one can check that if $f \in C_c^{\infty}(\mathbf{R}^{d+1})$ then for any multi-index α ,

$$\sup_{s, \xi} |\xi^{\alpha} \hat{f}(s, \xi)| = \sup_{s, \xi} |\mathcal{F}(D^{\alpha} f(s, \cdot))(\xi)| < \infty.$$

Therefore $\sup_s |\hat{f}(s, \xi)| \in L_1(\mathbf{R}^d)$ and from (2.2),

$$\begin{aligned} \int_{-\infty}^t |K(t, s, \cdot) * f(s, \cdot)(x)| \, ds &= \int_{-\infty}^t |\mathcal{F}^{-1}(\hat{K}(t, s, \xi) \hat{f}(s, \xi))(x)| \, ds \\ &\leq \int_{-\infty}^t \int_{\mathbf{R}^d} H(t-s, \xi) |\hat{f}(s, \xi)| \, d\xi \, ds \\ &= \int_{\mathbf{R}^d} \left| \sup_s \hat{f}(s, \xi) \right| \left(\sup_{\xi} \int_0^{\infty} H(t, \xi) \, dt \right) \, d\xi < \infty. \end{aligned}$$

It follows that $\mathcal{G}f$ is well defined for functions $f \in C_c^{\infty}(\mathbf{R}^{d+1})$.

Theorems 2.4 and 2.6 below are our main results. The proofs of the theorems are given in Sections 4 and 5.

Theorem 2.4. *Let Assumptions 2.1 and 2.2 hold and $p \in [2, \infty)$. Then for any $f \in C_c^\infty(\mathbf{R}^{d+1})$ it holds that*

$$\|\mathcal{G}f\|_{BMO(\mathbf{R}^{d+1})} \leq N\|f\|_{L_\infty(\mathbf{R}^{d+1})} \quad (2.7)$$

and

$$\|\mathcal{G}f\|_{L_p(\mathbf{R}^{d+1})} \leq N\|f\|_{L_p(\mathbf{R}^{d+1})}, \quad (2.8)$$

where the constant N depends only on d, p , and the constants in the assumptions.

Next, we formulate the conditions on the pseudo-differential operators $A(t)$ such that the kernels $K(t, s, x)$ related to $A(t)$ satisfy Assumptions 2.1 and 2.2. Let $A(t)$ be an operator with the symbol $\psi(t, \xi)$, that is

$$\mathcal{F}(A(t)u)(\xi) = \psi(t, \xi)\mathcal{F}(u)(\xi), \quad \forall u \in C_c^\infty(\mathbf{R}^d).$$

Define the kernel $p(t, s, x)$ by the formula

$$p(t, s, x) = I_{s < t} \mathcal{F}^{-1} \left(\exp \left(\int_s^t \psi(r, \xi) dr \right) \right) (x),$$

so that the solution of the equation

$$\frac{\partial u}{\partial t} = A(t)u + f, \quad t \in \mathbf{R}$$

is (formally) given by

$$u(t) = \int_{-\infty}^t (p(t, s, \cdot) * f(s, \cdot))(x) ds.$$

Denote

$$K(t, s, x) = (-\Delta)^{\gamma/2} p(t, s, x)$$

and

$$\mathcal{G}f(t, x) := (-\Delta)^{\gamma/2} u := \int_{-\infty}^t K(t, s, \cdot) * f(s, \cdot)(x) ds.$$

By $\Re z$ we denote the real part of z .

Assumption 2.5. *There exists a constant $\nu > 0$ such that for any $t \in \mathbf{R}, \xi \in \mathbf{R}^d \setminus \{0\}$ and multi-index $|\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1$,*

$$\Re[\psi(t, \xi)] \leq -\nu|\xi|^\gamma, \quad |D^\alpha \psi(t, \xi)| \leq \nu^{-1}|\xi|^{\gamma-|\alpha|}. \quad (2.9)$$

Theorem 2.6. *Let Assumption 2.5 hold and $p > 1$. Then*

(i) *for any $f \in C_c^\infty(\mathbf{R}^{d+1})$,*

$$\|\mathcal{G}f\|_{BMO(\mathbf{R}^{d+1})} \leq N(\nu, \gamma, d)\|f\|_{L_\infty(\mathbf{R}^{d+1})};$$

(ii) *for any $u \in C_c^\infty(\mathbf{R}^{d+1})$,*

$$\|u_t\|_{L_p(\mathbf{R}^{d+1})} + \|(-\Delta)^{\gamma/2} u\|_{L_p(\mathbf{R}^{d+1})} \leq N(p, \nu, \gamma, d)\|u_t - A(t)u\|_{L_p(\mathbf{R}^{d+1})}. \quad (2.10)$$

3. SOME FUNDAMENTAL ESTIMATES

In this section we estimate the mean oscillation of $\mathcal{G}f$ in terms of $\|f\|_{L_\infty}$. Recall that

$$\mathcal{G}f(t, x) := \int_{-\infty}^t K(t, s, \cdot) * f(s, \cdot)(x) \, ds.$$

We first derive an L_2 estimate of $\mathcal{G}f$.

Lemma 3.1. *Suppose that Assumption 2.1 holds and $f \in C_c^\infty(\mathbf{R}^{d+1})$. Then*

$$\|\mathcal{G}f\|_{L_2(\mathbf{R}^{d+1})} \leq N \|f\|_{L_2(\mathbf{R}^{d+1})},$$

where the constant N is independent of f . Consequently, the map $f \rightarrow \mathcal{G}f$ is extendable to a bounded linear operator on $L_2(\mathbf{R}^{d+1})$.

Proof. By Parseval's identity,

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{\mathbf{R}^d} |\mathcal{G}f(t, x)|^2 dx dt \\ &= N \int_{-\infty}^{\infty} \int_{\mathbf{R}^d} \left| \int_{-\infty}^t \hat{K}(t, s, \xi) \hat{f}(s, \xi) \, ds \right|^2 d\xi dt \\ &\leq N \int_{-\infty}^{\infty} \int_{\mathbf{R}^d} \left| \int_{-\infty}^{\infty} I_{s < t} |\hat{K}(t, s, \xi)| |\hat{f}(s, \xi)| \, ds \right|^2 d\xi dt. \end{aligned}$$

Hence it follows from Assumption 2.1 and Parseval's identity that

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{\mathbf{R}^d} |\mathcal{G}f(t, x)|^2 dx dt \\ &\leq N \int_{-\infty}^{\infty} \int_{\mathbf{R}^d} \left| \int_{-\infty}^{\infty} I_{s < t} H(t - s, \xi) |\hat{f}(s, \xi)| \, ds \right|^2 d\xi dt \\ &= N \int_{-\infty}^{\infty} \int_{\mathbf{R}^d} \left| \int_{-\infty}^{\infty} e^{it\tau} \int_{-\infty}^{\infty} I_{s < t} H(t - s, \xi) |\hat{f}(s, \xi)| \, ds dt \right|^2 d\xi d\tau \\ &= N \int_{-\infty}^{\infty} \int_{\mathbf{R}^d} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it\tau} I_{s < t} H(t - s, \xi) dt |\hat{f}(s, \xi)| \, ds \right|^2 d\xi d\tau \\ &\leq N \int_{-\infty}^{\infty} \int_{\mathbf{R}^d} \left| \int_0^{\infty} e^{it\tau} H(t, \xi) dt \right|^2 \left| \int_{\mathbf{R}} e^{is\tau} |\hat{f}(s, \xi)| \, ds \right|^2 d\xi d\tau \\ &\leq N \int_{-\infty}^{\infty} \int_{\mathbf{R}^d} \left| \int_{\mathbf{R}} e^{is\tau} |\hat{f}(s, \xi)| \, ds \right|^2 d\xi d\tau \\ &= N \int_{-\infty}^{\infty} \int_{\mathbf{R}^d} |\hat{f}(s, \xi)|^2 d\xi ds = N \int_{-\infty}^{\infty} \int_{\mathbf{R}^d} |f(s, x)|^2 dx ds. \end{aligned}$$

The lemma is proved. \square

For the rest of this section, \mathcal{G} is understood as a bounded linear operator on $L_2(\mathbf{R}^{d+1})$.

Corollary 3.2. *Let $f \in L_2(\mathbf{R}^{d+1})$ and vanish on $\mathbf{R}^{d+1} \setminus Q_{3c}(t_0, 0)$. Suppose that Assumption 2.1 holds. Then*

$$\int_{Q_c(t_0, 0)} |\mathcal{G}f(s, y)| \, ds dy \leq N |Q_c| \cdot \sup_{Q_{3c}(t_0, 0)} |f|,$$

where N does not depend on c, t_0 and f .

Proof. By Hölder's inequality and Lemma 3.1,

$$\begin{aligned}
\int_{Q_c(t_0,0)} |\mathcal{G}f(s,y)| \, dsdy &\leq \left(\int_{Q_c(t_0,0)} |\mathcal{G}f(s,y)|^2 \, dsdy \right)^{1/2} |Q_c|^{1/2} \\
&\leq \left(\int_{\mathbf{R}^{d+1}} |\mathcal{G}f(s,y)|^2 \, dsdy \right)^{1/2} |Q_c|^{1/2} \\
&\leq \left(\int_{\mathbf{R}^{d+1}} |f(s,y)|^2 \, dsdy \right)^{1/2} |Q_c|^{1/2} \\
&= \left(\int_{Q_{3c}(t_0,0)} |f(s,y)|^2 \, dsdy \right)^{1/2} |Q_c|^{1/2} \\
&\leq N|Q_c| \sup_{Q_{3c}(t_0,0)} |f|.
\end{aligned}$$

The lemma is proved. \square

In the following lemma we estimate the mean oscillation of $\mathcal{G}f$ on $Q_c(t_0,0)$ when f vanishes near $Q_c(t_0,0)$.

Lemma 3.3. *Suppose that Assumption 2.2 holds. Let $f \in L_2(\mathbf{R}^{d+1})$ and $f = 0$ on $Q_{2c}(t_0,0)$. Then*

$$\int_{Q_c(t_0,0)} \int_{Q_c(t_0,0)} |\mathcal{G}f(s,y) - \mathcal{G}f(r,z)| \, dsdrdydz \leq N|Q_c|^2 \cdot \sup_{\mathbf{R}^{d+1}} |f|, \quad (3.1)$$

where N does not depend on c, t_0 and f .

Proof. First we assume $f \in C_c^\infty(\mathbf{R}^{d+1})$. We will prove

$$\int_{Q_c(t_0,0)} |\mathcal{G}f(s,y) - \mathcal{G}f(t_0 - c^\gamma, 0)| \, dsdy \leq N|Q_c| \cdot \sup_{\mathbf{R}^{d+1}} |f|. \quad (3.2)$$

Let $(s,y) \in Q_c(t_0,0)$. Then

$$\begin{aligned}
&|\mathcal{G}f(s,y) - \mathcal{G}f(t_0 - c^\gamma, 0)| \\
&\leq |\mathcal{G}f(s,y) - \mathcal{G}f(s,0)| + |\mathcal{G}f(s,0) - \mathcal{G}f(t_0 - c^\gamma, 0)| \\
&=: \mathcal{I}_1 + \mathcal{I}_2
\end{aligned}$$

We consider \mathcal{I}_1 first.

$$\begin{aligned}
\mathcal{I}_1 &= \left| \int_{-\infty}^s \int_{\mathbf{R}^d} (K(s,\tau,y-z) - K(s,\tau,-z)) f(\tau,z) \, dzd\tau \right| \\
&= \left| \int_{t_0-(2c)^\gamma}^s \int_{\mathbf{R}^d} \cdots \, dzd\tau + \int_{-\infty}^{t_0-(2c)^\gamma} \int_{\mathbf{R}^d} \cdots \, dzd\tau \right| \\
&\leq \int_{t_0-(2c)^\gamma}^s \int_{\mathbf{R}^d} |K(s,\tau,z)| |f(\tau,y-z)| \, dzd\tau \\
&\quad + \int_{t_0-(2c)^\gamma}^s \int_{\mathbf{R}^d} |K(s,\tau,z)| |f(\tau,-z)| \, dz \\
&\quad + \int_{-\infty}^{t_0-(2c)^\gamma} \int_{\mathbf{R}^d} |K(s,\tau,y-z) - K(s,\tau,-z)| |f(\tau,z)| \, dzd\tau \\
&=: \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}.
\end{aligned}$$

Note that if $t_0 - (2c)^\gamma < \tau \leq s \leq t_0 + c^\gamma$ and $|z| \leq c$, then

$$f(\tau, y - z) = 0 \text{ and } f(\tau, -z) = 0, \quad (3.3)$$

because $|y - z| \leq 2c$ and $|-z| \leq c$, and $f = 0$ on $Q_{2c}(t_0, 0)$. Hence by (2.3), $\mathcal{I}_{11} + \mathcal{I}_{12}$ is less than or equal to

$$\begin{aligned} & N \sup_{\mathbf{R}^{d+1}} |f| \int_{t_0 - (2c)^\gamma}^s \int_{|z| \geq c} |K(s, \tau, z)| \, dz d\tau \\ & \leq N \varphi([s - (t_0 - (2c)^\gamma)]c^{-\gamma}) \sup_{\mathbf{R}^{d+1}} |f| \leq N \sup_{\mathbf{R}^{d+1}} |f|. \end{aligned}$$

Also, by (2.5),

$$\begin{aligned} \mathcal{I}_{13} & \leq N \sup_{\mathbf{R}^{d+1}} |f| \int_{-\infty}^{t_0 - (2c)^\gamma} \int_{\mathbf{R}^d} |K(s, \tau, y - z) - K(s, \tau, -z)| \, dz d\tau \\ & \leq N \varphi(c(s - t_0 + (2c)^\gamma)^{-1/\gamma}) \sup_{\mathbf{R}^{d+1}} |f| \\ & \leq N \sup_{\mathbf{R}^{d+1}} |f|. \end{aligned}$$

Next, we consider \mathcal{I}_2 . Note that

$$\begin{aligned} \mathcal{I}_2 & = \left| \mathcal{G}f(s, 0) - \mathcal{G}f(t_0 - c^\gamma, 0) \right| \\ & = \left| \int_{-\infty}^s \int_{\mathbf{R}^d} K(s, \tau, z) f(\tau, -z) \, dz d\tau - \int_{-\infty}^{t_0 - c^\gamma} \int_{\mathbf{R}^d} K(t_0 - c^\gamma, \tau, z) f(\tau, -z) \, dz d\tau \right| \\ & \leq \left| \int_{-\infty}^s \int_{\mathbf{R}^d} K(s, \tau, z) f(\tau, -z) \, dz d\tau - \int_{-\infty}^{t_0 - c^\gamma} \int_{\mathbf{R}^d} K(s, \tau, z) f(\tau, -z) \, dz d\tau \right| \\ & \quad + \left| \int_{-\infty}^{t_0 - c^\gamma} \int_{\mathbf{R}^d} [K(s, \tau, z) - K(t_0 - c^\gamma, \tau, z)] f(\tau, -z) \, dz d\tau \right| \\ & =: \mathcal{I}_{21} + \mathcal{I}_{22}. \end{aligned}$$

Recall that $f = 0$ on $[t_0 - (2c)^\gamma, t_0 + (2c)^\gamma] \times B_{2c}$. So by (2.3)

$$\begin{aligned} \mathcal{I}_{21} & \leq \int_{t_0 - c^\gamma}^s \int_{\mathbf{R}^d} |K(s, \tau, z)| |f(\tau, -z)| \, dz d\tau \\ & \leq \sup_{\mathbf{R}^{d+1}} |f| \int_{t_0 - c^\gamma}^s \int_{|z| \geq c} |K(s, \tau, z)| \, dz d\tau \\ & \leq N \varphi([s - (t_0 - c^\gamma)]c^{-\gamma}) \sup_{\mathbf{R}^{d+1}} |f| \leq N \sup_{\mathbf{R}^{d+1}} |f|. \end{aligned}$$

Also,

$$\begin{aligned} \mathcal{I}_{22} & \leq \int_{t_0 - (2c)^\gamma}^{t_0 - c^\gamma} \int_{\mathbf{R}^d} |K(s, \tau, z) - K(t_0 - c^\gamma, \tau, z)| |f(\tau, -z)| \, dz d\tau \\ & \quad + \sup_{\mathbf{R}^{d+1}} |f| \int_{-\infty}^{t_0 - (2c)^\gamma} \int_{\mathbf{R}^d} |K(s, \tau, z) - K(t_0 - c^\gamma, \tau, z)| \, dz d\tau \\ & =: \mathcal{I}_{221} + \mathcal{I}_{222}. \end{aligned}$$

Recalling (3.3), by (2.3) we have

$$\begin{aligned} \mathcal{I}_{221} &\leq \sup_{\mathbf{R}^{d+1}} |f| \int_{t_0-(2c)^\gamma}^s \int_{|z| \geq c} |K(s, \tau, z)| dz d\tau \\ &\quad + \sup_{\mathbf{R}^{d+1}} |f| \int_{t_0-(2c)^\gamma}^{t_0-c^\gamma} \int_{|z| \geq c} |K(t_0-c^\gamma, \tau, z)| dz d\tau \\ &\leq N \sup_{\mathbf{R}^{d+1}} |f|. \end{aligned}$$

On the other hand, by (2.4), we obtain

$$\mathcal{I}_{222} \leq \varphi([s - (t_0 - c^\gamma)](2^\gamma - 1)^{-1} c^{-\gamma}) \sup_{\mathbf{R}^{d+1}} |f| \leq N \sup_{\mathbf{R}^{d+1}} |f|.$$

Hence (3.2) is proved and this obviously implies (3.1) for $f \in C_c^\infty(\mathbf{R}^{d+1})$.

Now we consider the general case, that is $f \in L_2(\mathbf{R}^{d+1})$. For given $\varepsilon > 0$ we choose a sequence of functions $f_n \in C_c^\infty(\mathbf{R}^{d+1})$ such that $f_n = 0$ on $Q_{(2-2\varepsilon)c}(t_0, 0)$, $\mathcal{G}f_n \rightarrow \mathcal{G}f$ (a.e.) and $\sup_{\mathbf{R}^{d+1}} |f_n| \leq \sup_{\mathbf{R}^{d+1}} |f|$. Then by Fatou's theorem,

$$\begin{aligned} &\int_{Q_{(1-\varepsilon)c}(t_0, 0)} \int_{Q_{(1-\varepsilon)c}(t_0, 0)} |\mathcal{G}f(s, y) - \mathcal{G}f(r, z)| ds dr dy dz \\ &\leq \liminf_{n \rightarrow \infty} \int_{Q_{(1-\varepsilon)c}(t_0, 0)} \int_{Q_{(1-\varepsilon)c}(t_0, 0)} |\mathcal{G}f_n(s, y) - \mathcal{G}f_n(r, z)| ds dr dy dz \\ &\leq N|Q_c|^2 \cdot \liminf_{n \rightarrow \infty} \sup_{\mathbf{R}^{d+1}} |f_n| \leq N|Q_c|^2 \cdot \sup_{\mathbf{R}^{d+1}} |f|. \end{aligned}$$

Since ε is arbitrary the lemma is proved. \square

We introduce a simple decomposition of f . For any $\lambda > 0$ set

$$f_{1,\lambda}(t, x) := f(t, x)I_{|f| > \lambda}, \quad f_{2,\lambda}(t, x) := f(t, x)I_{|f| \leq \lambda}.$$

The following lemma is a modified version of Marcinkiewicz's interpolation theorem. We provide a proof for the sake of completeness.

Lemma 3.4. *Let \mathcal{A} be a subadditive operator on $L_2(\mathbf{R}^{d+1}) \cap L_\infty(\mathbf{R}^{d+1})$ and $f \in L_2(\mathbf{R}^{d+1}) \cap L_\infty(\mathbf{R}^{d+1})$. Suppose that*

$$\|\mathcal{A}(f_{1,\lambda})\|_{L_2(\mathbf{R}^{d+1})} \leq N_1 \|f_{1,\lambda}\|_{L_2(\mathbf{R}^{d+1})} \quad (3.4)$$

and

$$\|\mathcal{A}(f_{2,\lambda})\|_{L_\infty(\mathbf{R}^{d+1})} \leq N_2 \|f_{2,\lambda}\|_{L_\infty(\mathbf{R}^{d+1})} \quad (3.5)$$

for all $\lambda > 0$. Then for $p \in (2, \infty)$ we have

$$\|\mathcal{A}f\|_{L_p(\mathbf{R}^{d+1})} \leq N \|f\|_{L_p(\mathbf{R}^{d+1})},$$

where N depends only on d, p, N_1 , and N_2 .

Proof. Note that by Fubini's theorem

$$\|\mathcal{A}f\|_{L_p(\mathbf{R}^{d+1})}^p = N \int_0^\infty |\{(t, x) : |\mathcal{A}f(t, x)| > 2N_2\lambda\}| \lambda^{p-1} d\lambda. \quad (3.6)$$

Since for each $\lambda > 0$, $f = f_{1,\lambda} + f_{2,\lambda}$ and \mathcal{A} is subadditive,

$$\begin{aligned} &|\{(t, x) : |\mathcal{A}f(t, x)| > 2N_2\lambda\}| \\ &\leq |\{(t, x) : |\mathcal{A}f_{1,\lambda}(t, x)| > N_2\lambda\}| + |\{(t, x) : |\mathcal{A}f_{2,\lambda}(t, x)| > N_2\lambda\}|. \end{aligned}$$

Due to (3.5),

$$\|\mathcal{A}(f_{2,\lambda})\|_{L_\infty(\mathbf{R}^{d+1})} \leq N_2 \|f_{2,\lambda}\|_{L_\infty(\mathbf{R}^{d+1})} \leq N_2 \lambda,$$

which clearly implies

$$|\{(t, x) : |\mathcal{A}f_{2,\lambda}(t, x)| > N_2 \lambda\}| = 0.$$

Moreover by (3.4) and Chebyshev's inequality,

$$|\{(t, x) : |\mathcal{A}f_{1,\lambda}(t, x)| > N_2 \lambda\}| \leq N \frac{1}{\lambda^2} \|f_{1,\lambda}\|_{L_2(\mathbf{R}^{d+1})}^2.$$

Hence going back to (3.6), we get

$$\begin{aligned} \|\mathcal{A}f\|_{L_p(\mathbf{R}^{d+1})}^p &\leq N \int_0^\infty \frac{1}{\lambda^2} \|f_{1,\lambda}\|_{L_2(\mathbf{R}^{d+1})}^2 \lambda^{p-1} d\lambda \\ &\leq N \int_{\mathbf{R}^{d+1}} |f(t, x)|^2 \int_0^\infty I_{|f|>\lambda} \lambda^{p-3} d\lambda dt dx \\ &\leq N \int_{\mathbf{R}^{d+1}} |f(t, x)|^p dt dx. \end{aligned}$$

The lemma is proved. \square

4. PROOF OF THEOREM 2.4

Part I. We first prove (2.7) for $f \in L_2(\mathbf{R}^{d+1}) \cap L_\infty(\mathbf{R}^{d+1})$. It suffices to prove that for each $Q = Q_c(t_0, x_0)$

$$\oint_Q |\mathcal{G}f - (\mathcal{G}f)_Q| ds dy \leq N \sup_{\mathbf{R}^{d+1}} |f|.$$

Moreover, since $\mathcal{G}f(\cdot, \cdot)(t, x + x_0) = \mathcal{G}f(\cdot, x_0 + \cdot)(t, x)$, considering a translation we may assume that $x_0 = 0$. Thus

$$Q = Q_c(t_0, x_0) = (t_0 - c^\gamma, t_0 + c^\gamma) \times B_c(0).$$

Take $\zeta \in C_c^\infty(\mathbf{R}^{d+1})$ such that $\zeta = 1$ on Q_{2c} and $\zeta = 0$ outside of Q_{3c} . Then

$$\begin{aligned} &\oint_Q |\mathcal{G}f - (\mathcal{G}f)_Q| ds dy \\ &\leq 2 \oint_Q |\mathcal{G}(f\zeta)| ds dy + \oint_Q \oint_Q |\mathcal{G}(f(1-\zeta))(s, y) - \mathcal{G}(f(1-\zeta))(r, z)| ds dr dy dz \\ &=: \mathcal{I}_1 + \mathcal{I}_2. \end{aligned}$$

Due to Corollary 3.2,

$$\mathcal{I}_1 \leq N \frac{1}{|Q|} \int_Q |\mathcal{G}(f\zeta)(s, y)| ds dy \leq N \sup_{\mathbf{R}^{d+1}} |f\zeta| \leq N \sup_{\mathbf{R}^{d+1}} |f|.$$

On the other hand, by Lemma 3.3 we have

$$\mathcal{I}_2 \leq N \sup_{\mathbf{R}^{d+1}} |f(1-\zeta)| \leq N \sup_{\mathbf{R}^{d+1}} |f|.$$

Hence for any $f \in L_2(\mathbf{R}^{d+1}) \cap L_\infty(\mathbf{R}^{d+1})$ we have

$$\|\mathcal{G}f\|_{BMO(\mathbf{R}^{d+1})} \leq N \|f\|_{L_\infty(\mathbf{R}^{d+1})}, \quad (4.1)$$

where N is independent of f . Therefore (2.7) is proved.

Part II. Next we prove (2.8). For a measurable function $h(t, x)$ on \mathbf{R}^{d+1} , we define the maximal function

$$\mathcal{M}h(t, x) = \sup_Q \frac{1}{|Q|} \int_Q |f(r, z)| \, dr dz,$$

and the sharp function $h^\sharp(t, x)$

$$h^\sharp(t, x) = \sup_Q \frac{1}{|Q|} \int_Q |f(r, z) - f_Q| \, dr dz,$$

where $f_Q := \frac{1}{|Q|} \int_Q f(r, z) \, dr dz$, and the sup is taken all Q containing (t, x) of the type

$$Q = Q_c(t_0, x_0) := (t_0 - c^\gamma, t_0 + c^\gamma) \times B_c(x_0), \quad c > 0, \quad (t_0, x_0) \in \mathbf{R}^{d+1}.$$

Then by Fefferman-Stein theorem [10, Theorem 4.2.2], for any $h \in L_p(\mathbf{R}^{d+1})$,

$$\|h\|_{L_p(\mathbf{R}^{d+1})} \leq N \|h^\sharp\|_{L_p(\mathbf{R}^{d+1})}.$$

Moreover, by Hardy-Littlewood maximal theorem and the inequality $|h^\sharp(t, x)| \leq 2\mathcal{M}h(t, x)$,

$$\|h^\sharp\|_{L_p(\mathbf{R}^{d+1})} \leq N \|\mathcal{M}h\|_{L_p(\mathbf{R}^{d+1})} \leq N \|h\|_{L_p(\mathbf{R}^{d+1})}. \quad (4.2)$$

Combining Lemma 3.1 with (4.2), we get for any $f \in L_2(\mathbf{R}^{d+1})$,

$$\|(\mathcal{G}f)^\sharp\|_{L_2(\mathbf{R}^{d+1})} \leq N \|f\|_{L_2(\mathbf{R}^{d+1})}.$$

Moreover by (4.1),

$$\|(\mathcal{G}f)^\sharp\|_{L_\infty(\mathbf{R}^{d+1})} \leq N \|f\|_{L_\infty(\mathbf{R}^{d+1})}. \quad (4.3)$$

Note that the map $f \rightarrow (\mathcal{G}f)^\sharp$ is subadditive since \mathcal{G} is a linear operator. Hence by Lemma 3.4 for any $p \in [2, \infty)$ there exists a constant N such that

$$\|(\mathcal{G}f)^\sharp\|_{L_p(\mathbf{R}^{d+1})} \leq N \|f\|_{L_p(\mathbf{R}^{d+1})}, \quad \forall f \in L_2(\mathbf{R}^{d+1}) \cap L_\infty(\mathbf{R}^{d+1}).$$

Finally by Fefferman-Stein theorem, we get

$$\|\mathcal{G}f\|_{L_p(\mathbf{R}^{d+1})} \leq N \|f\|_{L_p(\mathbf{R}^{d+1})},$$

where N is independent of f . Therefore (2.8) is proved.

5. PROOF OF THEOREM 2.6

Recall that $A(t)$ is a pseudo differential operator with the symbol $\psi(t, \xi)$ satisfying

$$\Re[\psi(t, \xi)] \leq -\nu|\xi|^\gamma, \quad |D^\alpha \psi(t, \xi)| \leq \nu^{-1}|\xi|^{\gamma-|\alpha|}$$

for any multi-index $|\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1$. Also recall $p(t, s, x)$ and $K(t, s, x)$ are defined by

$$p(t, s, x) = I_{s < t} \mathcal{F}^{-1} \left(\exp \left(\int_s^t \psi(r, \xi) dr \right) \right) (x), \quad K(t, s, x) = (-\Delta)^{\gamma/2} p(t, s, x).$$

In this section we prove that $K(t, s, x)$ satisfies Assumptions 2.1 and 2.2 using the following auxiliary results.

Lemma 5.1. *Let $h \in C^2(\mathbf{R}^d \setminus \{0\})$ satisfy*

$$|h(x)| \leq N_0 |x|^\varsigma e^{-c|x|^\gamma}, \quad \forall x \in \mathbf{R}^d \setminus \{0\}, \quad (5.1)$$

with some constants $c, N_0 > 0$, $\varsigma > \eta - \frac{d}{2}$ and $\gamma > 0$. Further assume that either

$$\eta \in [0, 1) \quad \text{and} \quad |Dh(x)| \leq N_0 |x|^{\varsigma-1} e^{-c|x|^\gamma}, \quad \forall x \in \mathbf{R}^d \setminus \{0\} \quad (5.2)$$

or

$$\eta \in [1, 2) \quad \text{and} \quad |D^2h(x)| \leq N_0 |x|^{\varsigma-2} e^{-c|x|^\gamma}, \quad \forall x \in \mathbf{R}^d \setminus \{0\} \quad (5.3)$$

holds. Then

$$\|(-\Delta)^{\eta/2} h\|_{L_2(\mathbf{R}^d)} < N < \infty,$$

where $N = N(N_0, \eta, c, \varsigma, \gamma)$.

Proof. We assume $\eta \in (0, 2)$ since the statement is obvious if $\eta = 0$. We further assume $\varsigma < \eta$ because if (5.1)-(5.3) hold for some ς then they hold for any $\varsigma' \leq \varsigma$ (with other constant N_0).

Case 1. Suppose (5.2) holds. Let $C = C(\eta) > 0$ be the constant such that

$$-(-\Delta)^{\eta/2} h(x) = C \lim_{\varepsilon \rightarrow 0} \int_{|y| \geq \varepsilon} \frac{h(x+y) - h(x)}{|y|^{d+\eta}} dy = \mathcal{I}(x) + \mathcal{J}(x),$$

where

$$\mathcal{I}(x) = C \int_{|y| \geq |x|/2} \frac{h(x+y) - h(x)}{|y|^{d+\eta}} dy$$

and

$$\mathcal{J}(x) = C \lim_{\varepsilon \rightarrow 0} \int_{|x|/2 > |y| \geq \varepsilon} \frac{h(x+y) - h(x)}{|y|^{d+\eta}} dy.$$

Obviously,

$$|\mathcal{I}(x)| \leq C \int_{|y| \geq |x|/2} \frac{|h(x+y)|}{|y|^{d+\eta}} dy + C \int_{|y| \geq |x|/2} \frac{|h(x)|}{|y|^{d+\eta}} dy =: \mathcal{I}_1(x) + \mathcal{I}_2(x).$$

Recall $\eta > \varsigma$. From (5.1), if $|x| < 1$

$$\begin{aligned} \mathcal{I}_1 &\leq C \int_{|y| \geq |x|/2} \frac{|x+y|^\varsigma}{|y|^{d+\eta}} dy = C|x|^{-\eta} \int_{|y| \geq 1/2} \frac{|x+|x|y|^\varsigma}{|y|^{d+\eta}} dy \\ &\leq C|x|^{\varsigma-\eta} \sup_{|w|=1} \int_{|y| \geq 1/2} \frac{|w+y|^\varsigma}{|y|^{d+\eta}} dy \\ &\leq N|x|^{\varsigma-\eta}, \end{aligned} \quad (5.4)$$

where the last inequality is from the condition $\varsigma > \eta - d/2 > -d$. On the other hand, if $|x| \geq 1$ (recall $\varsigma > -d$ and $\gamma > 0$)

$$\begin{aligned} \mathcal{I}_1 &\leq C \int_{|y| \geq |x|/2} \frac{|x+y|^\varsigma e^{-c|x+y|^\gamma}}{|y|^{d+\eta}} dy \\ &\leq N \frac{1}{|x|^{d+\eta}} \int_{\mathbf{R}^d} |y|^\varsigma e^{-c|y|^\gamma} dy \leq N \frac{1}{|x|^{d+\eta}}. \end{aligned} \quad (5.5)$$

Also, using (5.1) again, we get

$$\mathcal{I}_2 \leq N|x|^{\varsigma-\eta} e^{-c|x|^\gamma}, \quad \forall x \in \mathbf{R}^d. \quad (5.6)$$

To estimate \mathcal{J} , we use Taylor's theorem and get

$$|\mathcal{J}| \leq N \int_{|y| < |x|/2} |\nabla h(x + \bar{\theta}y)| \frac{1}{|y|^{d-1+\eta}} dy,$$

where $0 \leq \bar{\theta} \leq 1$. So from (5.2),

$$|\mathcal{J}| \leq N|x|^{\varsigma-\eta} e^{-c(\frac{1}{2})^\gamma |x|^\gamma}, \quad \forall x \in \mathbf{R}^d. \quad (5.7)$$

Therefore by (5.4), (5.5), (5.6), and (5.7), we have

$$\int_{\mathbf{R}^d} |(-\Delta)^{n/2} h(x)|^2 dx < \infty$$

because $\varsigma > \eta - \frac{d}{2}$ and $\gamma > 0$.

Case 2. Suppose (5.3) holds. The proof for this case is very close to Case 1. It is enough to repeat the above proof, but in order to estimate \mathcal{J} we use the second order Taylor's theorem in stead of the first order one. \square

Before going further, for the simplicity of presentation we define

$$q_1(t, s, x) = I_{s < t} \mathcal{F}^{-1} \left(\exp \left(\int_s^t \psi(r, (t-s)^{-1/\gamma} \xi) dr \right) \right) (x),$$

and

$$\begin{aligned} q_2(t, s, x) \\ = (t-s) I_{s < t} \mathcal{F}^{-1} \left(\psi(t, (t-s)^{-1/\gamma} \xi) |\xi|^\gamma \exp \left(\int_s^t \psi(r, (t-s)^{-1/\gamma} \xi) dr \right) \right) (x). \end{aligned}$$

There are following relations among p , q_1 , and q_2 :

$$\begin{aligned} (t-s)^{d/\gamma} p(t, s, (t-s)^{1/\gamma} x) &= q_1(t, s, x), \\ (t-s)^{d/\gamma} (t-s) \Delta^{\gamma/2} p(t, s, (t-s)^{1/\gamma} x) &= \Delta^{\gamma/2} q_1(t, s, x), \end{aligned} \quad (5.8)$$

and

$$\frac{\partial}{\partial t} \Delta^{\gamma/2} p(t, s, x) = (t-s)^{-d/\gamma} (t-s)^{-2} q_2(t, s, (t-s)^{-1/\gamma} x). \quad (5.9)$$

These kernels have uniform upper bounds.

Lemma 5.2. *It holds that*

$$\sup_{t > s, x} |\Delta^{\gamma/2} q_1(t, s, x)| < \infty,$$

$$\sup_{t > s, x} \left| \frac{\partial}{\partial x^i} \Delta^{\gamma/2} q_1(t, s, x) \right| < \infty,$$

and

$$\sup_{t > s, x} |q_2(t, s, x)| < \infty.$$

Proof. From the properties of the Fourier transform, these are easy consequences of (2.9). The lemma is proved. \square

Lemma 5.3. *Let $0 < \delta < \left(\frac{1}{2} \wedge \gamma\right)$. Then*

$$\sup_{t>s} \int_{\mathbf{R}^d} \left| |x|^{\frac{d}{2}+\delta} |\Delta^{\gamma/2} q_1(t, s, x)| \right|^2 dx < \infty, \quad (5.10)$$

$$\sup_{t>s} \int_{\mathbf{R}^d} \left| |x|^{\frac{d}{2}+\delta} \left| \frac{\partial}{\partial x^i} \Delta^{\gamma/2} q_1(t, s, x) \right| \right|^2 dx < \infty, \quad (5.11)$$

and

$$\sup_{t>s} \int_{\mathbf{R}^d} \left| |x|^{\frac{d}{2}+\delta} |q_2(t, s, x)| \right|^2 dx < \infty. \quad (5.12)$$

Proof. First we prove (5.10). Let $t > s$. By Parseval's identity,

$$\begin{aligned} & \int_{\mathbf{R}^d} \left| |x|^{\frac{d}{2}+\delta} |\Delta^{\gamma/2} q_1(t, s, x)| \right|^2 dx \\ &= N \int_{\mathbf{R}^d} \left| \Delta^{\frac{d}{4}-\lfloor \frac{d}{4} \rfloor + \frac{\delta}{2}} \Delta^{\lfloor \frac{d}{4} \rfloor} \left(|\xi|^\gamma \exp \left(\int_s^t \psi(r, (t-s)^{-1/\gamma} \xi) dr \right) \right) \right|^2 dx. \end{aligned}$$

We apply Lemma 5.1 with

$$\eta = d/2 - 2\lfloor \frac{d}{4} \rfloor + \delta, \quad \varsigma = \gamma - 2\lfloor \frac{d}{4} \rfloor, \quad c = \nu$$

and

$$h(x) = \Delta^{\lfloor \frac{d}{4} \rfloor} \left(|\xi|^\gamma \exp \left(\int_s^t \psi(r, (t-s)^{-1/\gamma} \xi) dr \right) \right).$$

Note that since $\gamma > \delta$, we have $\varsigma > \eta - d/2$. Also, by (2.9),

$$\left| \Delta^{\lfloor \frac{d}{4} \rfloor} \left(|\xi|^\gamma \exp \left(\int_s^t \psi(r, (t-s)^{-1/\gamma} \xi) dr \right) \right) \right| \leq N |\xi|^{\gamma-2\lfloor \frac{d}{4} \rfloor} e^{-\nu|\xi|^\gamma}.$$

Thus (5.1) is satisfied with the above setting.

One can easily check that

$$\eta \in \begin{cases} [0, 1), & \text{if } d = 4k, 4k+1 \text{ for some integer } k \\ [1, 2), & \text{otherwise} \end{cases}$$

Therefore it is enough to prove (5.2) if $d = 4k$ or $4k+1$ for some integer k and (5.3) for the other case. These are easy consequences of (2.9), that is, we have

$$\left| D^1 \Delta^{\lfloor \frac{d}{4} \rfloor} \left(|\xi|^\gamma \exp \left(\int_s^t \psi(r, (t-s)^{-1/\gamma} \xi) dr \right) \right) \right| \leq N |\xi|^{\gamma-2\lfloor \frac{d}{4} \rfloor-1} e^{-\nu|\xi|^\gamma},$$

and

$$\left| D^2 \Delta^{\lfloor \frac{d}{4} \rfloor} \left(|\xi|^\gamma \exp \left(\int_s^t \psi(r, (t-s)^{-1/\gamma} \xi) dr \right) \right) \right| \leq N |\xi|^{\gamma-2\lfloor \frac{d}{4} \rfloor-2} e^{-\nu|\xi|^\gamma}.$$

Hence (5.10) is proved.

Both (5.11) and (5.12) can be proved similarly. We only remark main differences. Due to (2.9), for any $i = 1, 2, 3, \dots, d$ and multi-index $|\beta| \leq \lfloor \frac{d}{2} \rfloor + 1$

$$\left| D^\beta \left(|\xi|^\gamma \xi^i \exp \left(\int_s^t \psi(r, (t-s)^{-1/\gamma} \xi) dr \right) \right) \right| \leq N |\xi|^{\gamma+1-|\beta|} e^{-\nu|\xi|^\gamma}$$

and

$$\begin{aligned} & \left| D^\beta \left(\psi(s, (t-s)^{-1/\gamma} \xi) |\xi|^\gamma \exp \left(\int_s^t \psi(r, (t-s)^{-1/\gamma} \xi) dr \right) \right) \right| \\ & \leq N(t-s)^{-1} |\xi|^{2\gamma-|\beta|} e^{-\nu|\xi|^\gamma}. \end{aligned}$$

Hence for (5.11) we apply Lemma 5.1 with

$$\eta = d/2 - 2\lfloor \frac{d}{4} \rfloor + \delta, \quad \varsigma = \gamma + 1 - 2\lfloor \frac{d}{4} \rfloor, \quad c = \nu$$

and

$$h(x) = \Delta^{\lfloor \frac{d}{4} \rfloor} \left(|\xi|^\gamma \xi^i \exp \left(\int_s^t \psi(r, (t-s)^{-1/\gamma} \xi) dr \right) \right).$$

On the other hand for (5.12) we apply Lemma 5.1 with

$$\eta = d/2 - 2\lfloor \frac{d}{4} \rfloor + \delta, \quad \varsigma = 2\gamma - 2\lfloor \frac{d}{4} \rfloor, \quad c = \nu$$

and

$$h(x) = \Delta^{\lfloor \frac{d}{4} \rfloor} \left((t-s) \psi(s, (t-s)^{-1/\gamma} \xi) |\xi|^\gamma \exp \left(\int_s^t \psi(r, (t-s)^{-1/\gamma} \xi) dr \right) \right).$$

We skip the details. The lemma is proved. \square

By making full use of above lemmas, we obtain kernel estimates for $\Delta^{\gamma/2} p(t, s, x)$.

Lemma 5.4. *There exist constant $N > 0$ and $\varepsilon \in (0, 1)$ such that for all $s > r$, and $c > 0$*

$$\int_r^s \int_{|z| \geq c} |\Delta^{\gamma/2} p(s, \tau, z)| dz d\tau \leq N(s-r)^\varepsilon c^{-\varepsilon\gamma}.$$

Proof. From (5.8),

$$\int_{|z| \geq c} |\Delta^{\gamma/2} p(s, \tau, z)| dz = (s-\tau)^{-1} \int_{(s-\tau)^{1/\gamma}|z| \geq c} |\Delta^{\gamma/2} q_1(s, \tau, z)| dz.$$

For $0 < \varepsilon < 1$, if $(s-\tau)^{1/\gamma}|z| \geq c$, then

$$(s-\tau)^{-1} \leq (s-\tau)^{-1+\varepsilon} \left(\frac{|z|}{c} \right)^{\varepsilon\gamma}.$$

Therefore

$$\int_r^s \int_{|z| \geq c} |\Delta^{\gamma/2} p(s, \tau, z)| dz d\tau \leq c^{-\varepsilon\gamma} \int_r^s (s-\tau)^{-1+\varepsilon} \int_{\mathbf{R}^d} |z|^{\varepsilon\gamma} |\Delta^{\gamma/2} q_1(s, \tau, z)| dz d\tau. \quad (5.13)$$

We claim

$$\sup_{s>\tau>0} \int_{\mathbf{R}^d} |z|^{\varepsilon\gamma} |\Delta^{\gamma/2} q_1(s, \tau, z)| dz < \infty.$$

By Lemma 5.2 and Hölder's inequality,

$$\begin{aligned}
& \int_{\mathbf{R}^d} |z|^{\varepsilon\gamma} |\Delta^{\gamma/2} q_1(s, \tau, z)| \, dz \\
& \leq \int_{|z|<1} |z|^{\varepsilon\gamma} |\Delta^{\gamma/2} q_1(s, \tau, z)| \, dz + \int_{|z|\geq 1} |z|^{\varepsilon\gamma} |\Delta^{\gamma/2} q_1(s, \tau, z)| \, dz \\
& \leq N + N \left(\int_{|z|\geq 1} |z|^{-d-\varepsilon\gamma} \, dz \right)^{1/2} \left(\int_{|z|\geq 1} \left| |z|^{\frac{d+3\varepsilon\gamma}{2}} |\Delta^{\gamma/2} q_1(s, \tau, z)| \right|^2 \, dz \right)^{1/2}.
\end{aligned}$$

Due to Lemma 5.3 (i) with small ε so that $\frac{3\varepsilon\gamma}{2} < \left(\frac{1}{2} \wedge \gamma\right)$,

$$\sup_{s>\tau} \int_{\mathbf{R}^d} \left| |z|^{\frac{d+3\varepsilon\gamma}{2}} |\Delta^{\gamma/2} q_1(s, \tau, z)| \right|^2 \, dz < \infty.$$

Therefore, the claim is proved. Going back to (5.13), we conclude that

$$\int_r^s \int_{|z|\geq c} |\Delta^{\gamma/2} p(s, \tau, z)| \, dz d\tau \leq N c^{-\varepsilon\gamma} \int_r^s (s-\tau)^{-1+\varepsilon} d\tau \leq c^{-\varepsilon\gamma} (s-r)^\varepsilon.$$

The lemma is proved. \square

Lemma 5.5. *There exist constants $N > 0$ and $\varepsilon \in (0, 1)$ such that for all $s > r > a$ and $h \in \mathbf{R}^d$*

$$\int_{-\infty}^a \int_{\mathbf{R}^d} |\Delta^{\gamma/2} p(s, \tau, z+h) - \Delta^{\gamma/2} p(s, \tau, z)| \, dz d\tau \leq N |h| (s-a)^{-1/\gamma} \quad (5.14)$$

and

$$\int_{-\infty}^a \int_{\mathbf{R}^d} |\Delta^{\gamma/2} p(s, \tau, z) - \Delta^{\gamma/2} p(r, \tau, z)| \, dz d\tau \leq N (s-r) (r-a)^{-1}.$$

Proof. First we show (5.14). From (5.8),

$$\frac{\partial}{\partial x^i} \Delta^{\gamma/2} p(s, \tau, z) = (s-\tau)^{-d/\gamma} (s-\tau)^{-1-1/\gamma} \frac{\partial}{\partial x^i} \Delta^{\gamma/2} q_1(s, \tau, (s-\tau)^{-1/\gamma} z). \quad (5.15)$$

Fix $0 < \delta < \left(\frac{1}{2} \wedge \gamma\right)$. Then by Hölder's inequality, Lemmas 5.2 and 5.3,

$$\begin{aligned}
& \sup_{s>\tau} \left((s-\tau)^{-d/\gamma} \int_{\mathbf{R}^d} \left| \frac{\partial}{\partial x^i} \Delta^{\gamma/2} q_1(s, \tau, (s-\tau)^{-1/\gamma} z) \right| \, dz \right) \\
& = \sup_{s>\tau} \int_{\mathbf{R}^d} \left| \frac{\partial}{\partial x^i} \Delta^{\gamma/2} q_1(s, \tau, z) \right| \, dz \\
& \leq N + \sup_{s>\tau} \left[\left(\int_{|z|\geq 1} |z|^{-d-2\delta} \, dz \right)^{1/2} \left(\int_{|z|\geq 1} \left| |z|^{d/2+\delta} \frac{\partial}{\partial x^i} \Delta^{\gamma/2} q_1(s, \tau, z) \right|^2 \, dz \right)^{1/2} \right] \\
& < \infty.
\end{aligned}$$

Therefore, by the mean-value theorem and (5.15)

$$\begin{aligned}
& \int_{-\infty}^a \int_{\mathbf{R}^d} |\Delta^{\gamma/2} p(s, \tau, z+h) - \Delta^{\gamma/2} p(s, \tau, z)| \, dz d\tau \\
& \leq |h| \int_{-\infty}^a \int_{\mathbf{R}^d} |\nabla \Delta^{\gamma/2} p(s, \tau, z)| \, dz d\tau \\
& \leq |h| \int_{-\infty}^a (s-\tau)^{-1-1/\gamma} \int_{\mathbf{R}^d} \left| \frac{\partial}{\partial x^i} \Delta^{\gamma/2} q_1(s, \tau, z) \right| \, dz d\tau \\
& \leq N|h| \int_{-\infty}^a (s-\tau)^{-1-1/\gamma} d\tau \leq N|h| \int_{s-a}^{\infty} \tau^{-1-1/\gamma} d\tau = N|h|(s-a)^{-1/\gamma}.
\end{aligned}$$

In order to prove the second assertion, observe that by the mean-value theorem and (5.9),

$$\begin{aligned}
& |\Delta^{\gamma/2} p(s, \tau, z) - \Delta^{\gamma/2} p(r, \tau, z)| \\
& \leq |s-r| \left| \frac{\partial}{\partial t} \Delta^{\gamma/2} p(\theta s + (1-\theta)r, \tau, z) \right| \\
& \leq |s-r| (\theta s + (1-\theta)r - \tau)^{-d/\gamma-2} |q_2(\theta s + (1-\theta)r, \tau, (\theta s + (1-\theta)r - \tau)^{-1/\gamma} z)|.
\end{aligned}$$

Following the proof of the first assertion with Lemma 5.3 (iii), we get

$$\sup_{s>\tau, r>\tau, 0 \leq \theta \leq 1} \int_{\mathbf{R}^d} |q_2(\theta s + (1-\theta)r, \tau, z)| \, dz < \infty.$$

Therefore,

$$\begin{aligned}
\int_{-\infty}^a \int_{\mathbf{R}^d} |\Delta^{\gamma/2} p(s, \tau, z) - \Delta^{\gamma/2} p(r, \tau, z)| \, dz d\tau & \leq \int_{-\infty}^a \frac{|s-r|}{(\theta s + (1-\theta)r - \tau)^2} d\tau \\
& \leq |s-r|(r-a)^{-1}.
\end{aligned}$$

The lemma is proved. \square

Proof of Theorem 2.6

From Lemma 5.4 and Lemma 5.5, it is proved that the kernel $K(s, \tau, z) := \Delta^{\gamma/2} p(s, \tau, z)$ satisfies Assumption 2.2. Moreover, by the definition of the kernel,

$$\begin{aligned}
\left| \mathcal{F} \left(\Delta^{\gamma/2} p(t, s, \cdot) \right) (\xi) \right| & \leq |\xi|^\gamma \left| \exp \left(\int_s^t \psi(r, \xi) dr \right) \right| \\
& \leq |\xi|^\gamma \exp \left(-\nu(t-s) |\xi|^\gamma \right)
\end{aligned}$$

where the second inequality is due to (2.9). Hence $K(s, \tau, z) = \Delta^{\gamma/2} p(s, \tau, z)$ also satisfies assumption 2.1 because obviously

$$\sup_{\xi} \int_0^\infty |\xi|^\gamma \exp \left(-\nu t |\xi|^\gamma \right) dt < \infty.$$

Therefore, due to Theorem 2.4, for any $p \geq 2$ it holds that

$$\|\mathcal{G}f\|_p \leq N\|f\|_p, \quad \forall f \in C_c^\infty(\mathbf{R}^{d+1}). \quad (5.16)$$

Since the operator $f \rightarrow \mathcal{G}f$ is linear and (5.16) holds for all $f \in C_c^\infty(\mathbf{R}^{d+1})$, the operator \mathcal{G} is extendible to a bounded linear operator on $L_p(\mathbf{R}^{d+1})$, and (5.16) holds for all $f \in L_p(\mathbf{R}^{d+1})$.

Now assume $u \in C_c^\infty(\mathbf{R}^{d+1})$. Denote $f := u_t - A(t)u$. Then obviously $f \in L_p(\mathbf{R}^{d+1})$. Thus to prove (2.10) we only need to show $(-\Delta)^{\gamma/2}u = \mathcal{G}f$. Taking the Fourier transform to the equation $u_t - A(t)u = f$, one easily gets

$$\hat{u}(t, \xi) = \int_{-\infty}^t e^{\int_s^t \psi(r, \xi) dr} \hat{f}(s, \xi) ds.$$

This and the inverse Fourier transform certainly lead to

$$u(t, x) = \int_{-\infty}^t p(t, s, \cdot) * f(s, \cdot)(x) ds, \quad (-\Delta)^{\gamma/2}u = \mathcal{G}f. \quad (5.17)$$

These equalities are because f has compact support and is sufficiently smooth with respect to x uniformly in t (cf. Remark 2.3).

Next we prove (2.10) for $p \in (1, 2)$ by using the duality argument. Let $q \in (2, \infty)$ be the conjugate of p . Consider the kernel

$$\begin{aligned} P(t, s, x) &= K(-s, -t, x) = I_{-t < -s} \mathcal{F}^{-1} \left\{ |\xi|^\gamma \exp \left(\int_{-t}^{-s} \psi(r, \xi) dr \right) \right\} \\ &= I_{t > s} \mathcal{F}^{-1} \left\{ |\xi|^\gamma \exp \left(\int_s^t \psi(-r, \xi) dr \right) \right\}. \end{aligned}$$

Note that $\psi(-t, \xi)$ also satisfies Assumption 2.5. Define operator \mathcal{P} by

$$\mathcal{P}h(t, x) := \int_{\mathbf{R}^{d+1}} P(t, s, x - y) h(s, y) dy ds.$$

Considering the change of variable $(s, t) \rightarrow (-s, -t)$, we observe that by Fubini's theorem, for $f, g \in C_c^\infty(\mathbf{R}^{d+1})$,

$$\begin{aligned} &\int_{\mathbf{R}^{d+1}} g(s, y) \mathcal{G}f(s, y) dy ds \\ &= \int_{\mathbf{R}^{d+1}} g(s, y) \left(\int_{\mathbf{R}^{d+1}} K(s, t, y - x) f(t, x) dx dt \right) dy ds \\ &= \int_{\mathbf{R}^{d+1}} f(-t, -x) \left(\int_{\mathbf{R}^{d+1}} K(-s, -t, y) g(-s, y - x) dy ds \right) dx dt \\ &= \int_{\mathbf{R}^{d+1}} f(-t, -x) \left(\int_{\mathbf{R}^{d+1}} P(t, s, y) \tilde{g}(s, x - y) dy ds \right) dx dt \\ &= \int_{\mathbf{R}^{d+1}} f(-t, -x) \mathcal{P}\tilde{g}(t, x) dx dt \end{aligned}$$

where $\tilde{g}(t, x) = g(-t, -x)$. Then by Hölder inequality and the fact that $2 < q < \infty$, we have

$$\left| \int_{\mathbf{R}^{d+1}} g(t, x) \mathcal{G}f(t, x) dx dt \right| \leq N \|f\|_{L_p} \|\mathcal{P}\tilde{g}\|_{L_q} \leq N \|f\|_{L_p} \|g\|_{L_q}.$$

Since $g \in C_c^\infty(\mathbf{R}^{d+1})$ is arbitrary, (5.16) is proved for $p \in (1, 2)$. Reminding (5.17), we obtain (2.10) for $p \in (1, 2)$. The theorem is proved.

6. APPLICATIONS

For applications of Theorem 2.6 we introduce $2m$ -order operator

$$A_1(t)u := (-1)^{m-1} \sum_{|\alpha|=|\beta|=m} a^{\alpha\beta}(t) D^{\alpha+\beta}u,$$

and γ -order (nonlocal) operator

$$A_2(t) := -a(t)(-\Delta)^{\gamma/2},$$

where the coefficients $a^{\alpha\beta}(t)$ and $a(t)$ are bounded complex-valued measurable functions satisfying

$$\nu < \Re[a(t)] < \nu^{-1},$$

and

$$\nu|\xi|^{2m} \leq \sum_{|\alpha|=|\beta|=m} \xi^\alpha \xi^\beta \Re[a^{\alpha\beta}(t)] \leq \nu^{-1}|\xi|^{2m}, \quad \forall \xi \in \mathbf{R}^d.$$

Corollary 6.1. *Let $p > 1$. Then for any $u \in C_c^\infty(\mathbf{R}^{d+1})$,*

$$\|u_t\|_{L_p(\mathbf{R}^{d+1})} + \|(-\Delta)^m u\|_{L_p(\mathbf{R}^{d+1})} \leq N\|u_t - A_1(t)u\|_{L_p(\mathbf{R}^{d+1})},$$

where N depends only on p, ν, m and d .

Proof. It is obvious that the symbol $\psi(t, \xi) = -a^{\alpha\beta}(t)\xi^\alpha \xi^\beta$ satisfies (2.9) with $\gamma = 2m$ and any multi-index α . Thus the corollary follows from Theorem 2.6. \square

Corollary 6.2. *Let $p > 1$. Then for any $u \in C_c^\infty(\mathbf{R}^{d+1})$,*

$$\|u_t\|_{L_p(\mathbf{R}^{d+1})} + \|(-\Delta)^{\gamma/2} u\|_{L_p(\mathbf{R}^{d+1})} \leq N\|u_t - A_2(t)u\|_{L_p(\mathbf{R}^{d+1})},$$

where N depends only on p, ν, γ and d .

Proof. The symbol related to the operator $A_2(t)$ is $-a(t)|\xi|^\gamma$, and therefore the corollary follows from Theorem 2.6. \square

Recall we defined $(-\Delta)^{\gamma/2}$ as the operator with symbol $|\xi|^\gamma$ for any $\gamma \in (0, \infty)$. For further applications of Theorem 2.6, we consider a product of $(-\Delta)^k$ and an integro-differential operator $\mathcal{L}_0 = \mathcal{L}_{0,\gamma}$. We remark that in place of $(-\Delta)^k$ one can consider many other pseudo-differential or high order differential operators.

Fix $\gamma \in (0, 2)$, and for $k = 0, 1, 2, \dots$ denote

$$\mathcal{L}_k(t)u = (-\Delta)^k \mathcal{L}_{0,\gamma} u$$

$$:= \int_{\mathbf{R}^d \setminus \{0\}} \left((-\Delta)^k u(t, x+y) - (-\Delta)^k u(t, x) - \chi(y)(\nabla(-\Delta)^k u(t, x), y) \right) \frac{m(t, y)}{|y|^{d+\gamma}} dy$$

where $\chi(y) = I_{\gamma>1} + I_{|y|\leq 1} I_{\gamma=1}$ and $m(t, y) \geq 0$ is a measurable function satisfying the following conditions :

(i) If $\gamma = 1$ then

$$\int_{\partial B_1} w m(t, w) S_1(dw) = 0, \quad \forall t > 0, \quad (6.1)$$

where ∂B_1 is the unit sphere in \mathbf{R}^d and $S_1(dw)$ is the surface measure on it.

(ii) The function $m = m(t, y)$ is zero-order homogeneous and differentiable in y up to $d_0 = \lfloor \frac{d}{2} \rfloor + 1$.

(iii) There is a constant K such that for each $t \in \mathbf{R}$

$$\sup_{|\alpha| \leq d_0, |y|=1} |D_y^\alpha m^{(\alpha)}(t, y)| \leq K.$$

It turns out that the operator \mathcal{L}_k is a pseudo differential operator with symbol

$$\psi(t, \xi) = -c_1 |\xi|^{2k} \int_{\partial B_1} |(w, \xi)|^\gamma [1 - i\varphi^{(\gamma)}(w, \xi)] m(t, w) S_1(dw),$$

$$\varphi^{(\gamma)}(w, \xi) = c_2 \frac{(w, \xi)}{|(w, \xi)|} I_{\gamma \neq 1} - \frac{2}{\pi} \frac{(w, \xi)}{|(w, \xi)|} \ln |(w, \xi)| I_{\gamma=1},$$

and $c_1(\gamma, d)$, $c_2(\gamma, d)$ are certain positive constants.

(iv) There is a constant $N_0 > 0$ such that the symbol $\psi(t, \xi)$ of \mathcal{L}_k satisfies

$$\sup_{t, |\xi|=1} \Re[\psi(t, \xi)] \leq -N_0. \quad (6.2)$$

One can check that (6.2) holds if there exists a constant $c > 0$ so that $m(t, y) > c$ on a set $E \subset \partial B_1$ of positive $S_1(dw)$ -measure.

Corollary 6.3. *Let $p > 1$ and $k = 0, 1, 2, \dots$. Then under above conditions (i)-(iv) on $m(t, y)$ it holds that for any $u \in C_c^\infty(\mathbf{R}^{d+1})$*

$$\|u_t\|_{L_p(\mathbf{R}^{d+1})} + \|(-\Delta)^{\gamma/2+k} u\|_{L_p(\mathbf{R}^{d+1})} \leq N \|u_t - \mathcal{L}_k u\|_{L_p(\mathbf{R}^{d+1})},$$

where N depends only on p, γ, k, d, N_0 and K .

Proof. Note that for $\xi \neq 0$

$$\psi(t, \xi) = |\xi|^{2k+\gamma} \psi\left(t, \frac{\xi}{|\xi|}\right) =: |\xi|^{2k+\gamma} \tilde{\psi}(t, \xi).$$

The above equality is obvious if $\gamma \neq 1$, and if $\gamma = 1$ then by (6.1)

$$\begin{aligned} \psi(t, \xi) &= |\xi|^{2k+1} \psi\left(t, \frac{\xi}{|\xi|}\right) + |\xi|^{2k} \ln |\xi| \int_{\partial B_1} (w, \xi) m(t, w) S_1(dw) \\ &= |\xi|^{2k+1} \psi\left(t, \frac{\xi}{|\xi|}\right). \end{aligned}$$

By using condition (iii) one can check (see e.g. [9, Remark 2.6]) that for any multi-index α , $|\alpha| \leq d_0$, there exists a constant $N = N(\alpha)$ such that

$$|D^\alpha \tilde{\psi}(t, \xi)| \leq N |\xi|^{-|\alpha|}.$$

Thus it is obvious that the given symbol ψ satisfies (2.9). The corollary is proved. \square

Next we discuss the issue regarding the compositions and powers of operators. Let $B_1(t)$ and $B_2(t)$ be linear operators with symbols $\psi_1(t)$ and $\psi_2(t)$ satisfying (2.9), that is there exist constants $\gamma_1, \gamma_2, \nu_1, \nu_2 > 0$ so that

$$\Re[-\psi_i(t, \xi)] \geq \nu_i |\xi|^{\gamma_i}, \quad |D^\alpha \psi_i(t, \xi)| \leq \nu_i^{-1} |\xi|^{\gamma_i - |\alpha|}, \quad (i = 1, 2),$$

for any multi-index α , $|\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1$. Fix $a, b > 0$, and denote $\gamma := a\gamma_1 + b\gamma_2$. Consider γ -order operator

$$C(t) = -(-A_1(t))^a (-A_2(t))^b$$

with the symbol $\psi = -(-\psi_1)^a (-\psi_2)^b$. It is easy to check that there exists a constant $N > 0$ so that for any multi-index α , $|\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1$,

$$|D^\alpha \psi(t, \xi)| \leq N |\psi|^{\gamma - |\alpha|}, \quad \xi \in \mathbf{R}^d \setminus \{0\}.$$

Therefore, Theorem 2.6 is applicable to the operator $C(t) = -(A_1(t))^a(A_2(t))^b$ if

$$\Re[-\psi(t, \xi)] = \Re[(-\psi_1)^a(-\psi_2)^b] \geq N^{-1}|\xi|^\gamma, \quad \forall \xi \in \mathbf{R}^d. \quad (6.3)$$

Obviously (6.3) is satisfied if, for instance, the symbols $\psi_i(t, \xi)$ are real-valued. In this case, for any $u \in C_c^\infty(\mathbf{R}^{d+1})$, we have

$$\|u_t\|_{L_p(\mathbf{R}^{d+1})} + \|(-\Delta)^{\gamma/2}u\|_{L_p(\mathbf{R}^{d+1})} \leq N\|u_t - C(t)u\|_{L_p(\mathbf{R}^{d+1})}.$$

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